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On pantograph multi-point boundary value problem with Caputo q-fractional derivative

Houari Bouzid 0000-0002-6242-7857,
Abdelkader Benali 0000-0001-6205-3499,
Abdelkrim Salim* 0000-0003-2795-6224

Abstract. The purpose of this research is to investigate the existence and uniqueness of several solutions to the multi-point boundary value problem of nonlinear fractional differential equations involving two fractional derivatives. We demonstrate the existence of solutions by applying a number of fixed point theorems, including Banach's fixed point theorem, nonlinear alternative of Leray-Schauder type, and Leray-Schauder degree. Finally, two examples are presented to demonstrate our results.

1. Introduction

Many physical processes, including charge transport in amorphous semiconductors [30], electrochemistry, and material science, give rise to fractional derivatives, which are in fact represented by differential equations of fractional order [1, 11–15, 19, 20, 24, 25]. Recent years have seen an increase in the number of papers on fractional differential equations that use various operators, including Riemann-Liouville operators [26, 34], Caputo operators [4, 5, 22, 35], Hadamard operators [32], and q-fractional operators [3].

In 1910, Frank Hilton Jackson pioneered the introduction and development of q-calculus by defining the q-analog of the ordinary derivative [31]. Given the significance of this theory, q-difference equations and operators have been thoroughly investigated and the definitions of the q-derivative (a modification of the classical derivative), q-integral, q-factorial and specific functions have been established by various researchers [3, 6, 10, 21, 23, 27–29].

Furthermore, many authors have obtained the existence and uniqueness of solutions for various classes of fractional differential equations by using various nonlinear analysis techniques. As an example, we recommend that the reader review the references listed in [4, 8, 9, 16–18, 26, 33].

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In [29], we considered the following fractional q-difference problem:

$$\begin{cases} ({}^cD_q^\zeta y)(t) = \wp(t, y(t));\ t \in f := [0, \beta], \\ y(0) = y_0 \in F, \end{cases}$$

where $q \in (0,1)$, $\zeta \in (0,1]$, $\beta > 0$, $\wp : f \times F \to F$ is a given continuous function, F is a real (or complex) Banach space with norm $\|\cdot\|$, and ${}^cD_q^\zeta$ is the Caputo fractional q-difference derivative of order ζ.

In [23], the authors proved some existence of solutions for the following problem with implicit fractional q-difference equations in Banach algebras:

$$\begin{cases} {}^cD_q^\zeta \left(\frac{y(t)}{h(t,y(t))}\right) = \psi\left(t, y(t), {}^cD_q^\zeta\left(\frac{y(t)}{h(t,y(t))}\right)\right);\ t \in f := [0, \beta], \\ y(0) = y_0 \in \mathbb{R}, \end{cases}$$

where $q \in (0,1)$, $\zeta \in (0,1]$, $\beta > 0$, $h : f \times \mathbb{R} \to \mathbb{R}^*$, $\psi : f \times \mathbb{R}^2 \to \mathbb{R}$ are given functions.

In [10], the authors considered the following problem:

$$\begin{cases} y(t) = g(t);\ t \in [-\varepsilon, 0], \\ {}^cD_q^\zeta(y(t) - \Upsilon(t, y_t)) = \wp(t, y(t), {}^cD_q^\zeta(y(t) - \Upsilon(t, y_t)));\ t \in f := [0, \beta], \end{cases}$$

where $q \in (0,1)$, $\zeta \in (0,1]$, $\beta, \varepsilon > 0$, $g \in \mho$, $\Upsilon : f \times \mho \to \mathbb{R}$, $\wp : f \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

In this research, we consider the problem:

$$\begin{cases} {}^C\mathrm{D}_q^\zeta y(t) = f\big(t, y(t), y(\lambda t)\big), \quad 0 < q < 1,\ t \in [0, beta], \\ y(0) = 0, \qquad \mathrm{D}_q y(\beta) = \sum\limits_{\varrho=1}^{m} \nu_\varrho \mathrm{I}_q^{\zeta-1} y(\eta_\varrho), \quad 0 < \eta_\varrho < \beta, \end{cases} \tag{1}$$

where ${}^C\mathrm{D}_q^\zeta$ is the fractional q-derivative of the Caputo type of orders $\zeta \in (1,2]$, D_q is the first q-derivative, $\mathrm{I}_q^{\zeta-1}$ is the Riemann-Liouville fractional q-integral of order $\zeta - 1 > 0$, $0 < \lambda, q < 1$, ν_ϱ are real constants for $1 \leq \varrho \leq m$, $m \geq 2$ and $f : [0, \beta] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous functions on $[0, \beta]$.

The following are the primary novelties of the current paper:

- Considering the diverse conditions we applied to problem (1), our work can be seen as a continuation of the studies mentioned above.
- Our findings expand upon those in [10, 23, 29] by introducing pantograph arguments and a boundary value problem with a nonlocal condition.
- To establish our results, we utilized several methods including fixed point theorems, and we also demonstrate that these outcomes can be achieved using the Leray-Schauder degree and Hölder inequality.

2. Preliminaries

Let $0 < q \neq 1$ and consider a q-real number $[a]_q = \frac{1-q^a}{1-q}$, for $a \in \mathbb{R}$. The q-analogue of the Pochhammer symbol (q-shifted factorial) is defined as

$$(a;q)_m = \begin{cases} 1, & m = 0, \\ \prod_{j=0}^{m-1}(1-aq^j), & m = 1,2,3,\ldots \end{cases}$$

The q-analogue of the exponent $(a-b)^m$ is expressed by, for $a, b \in \mathbb{R}$,

$$(a-b)_m = \begin{cases} 1, & m = 0, \\ \prod_{j=0}^{m-1}(a-bq^j), & m = 1,2,3,\ldots \end{cases}$$

The q-factorial is given by

$$[m]_q = \prod_{j=0}^{m-1}[r]_q = \frac{(q;q)_m}{(1-q)^m}, \quad m \in \mathbb{N}.$$

Definition 1 ([7]). The q-gamma function $\Gamma_q(\zeta)$ is defined as

$$\Gamma_q(\zeta) = \frac{(1-q)^{(\zeta-1)}}{(1-q)^{\zeta-1}} = (q;q)_{\zeta-1}(1-q)^{\zeta-1}, \quad \zeta \in \mathbb{C} \setminus \{-n : n \in \mathbb{N} \cup \{0\}\},$$

with $\Gamma_q(\zeta+1) = [\zeta]_q \Gamma_q(\zeta)$.

Definition 2 ([2]). For the given function μ which is defined on $[0,1]$, the Riemann-Liouville q-integral of fractional order $\zeta \geq 0$ is $(\mathrm{I}_q^0 \mu)(t) = \mu(t)$ and

$$\mathrm{I}_q^\zeta \mu(t) = \int_0^t \frac{(t-qs)^{(\zeta-1)}}{\Gamma_q(\zeta)} \mu(s)\, \mathrm{d}_q s = t^\zeta (1-q)^\zeta \sum_{m=0}^{\infty} q^m \frac{(q^\zeta;q)_m}{(q;q)_m} \mu(tq^m),$$

for $\zeta > 0$, $t \in [0,1]$.

Definition 3 ([7]). The Caputo fractional q-derivative of order $\zeta \in (n-1, n)$ of the continuos functionis $\mu : [0,\beta] \to \mathbb{R}$, denoted by ${}^{\mathrm{C}}\mathrm{D}_q^\zeta$ is defined by

$$({}^{\mathrm{C}}\mathrm{D}_q^\zeta \mu)(t) = \mathrm{I}_q^{[\zeta]-\zeta} \mathrm{D}_q^{[\zeta]} \mu(t).$$

Furthermore, the q-derivative of a function $\mu(t)$ is expressed as

$$(\mathrm{D}_q \mu)(t) = \frac{\mu(t) - \mu(qt)}{t - qt}, \quad t \neq 0; \qquad (\mathrm{D}_q \mu)(0) = \lim_{t \to 0} (\mathrm{D}_q \mu)(t).$$

Lemma 1 ([7]). *Let $\zeta > 0$ and $m \in \mathbb{N}$. Then,*

$$\mathrm{I}_q^\zeta {}^{\mathrm{C}}\mathrm{D}_q^\zeta \mu(t) = \mu(t) - \sum_{\varrho=0}^{[\zeta]} c_m t^m, \tag{2}$$

$$ {}^{\mathrm{C}}\mathrm{D}_q^\zeta \mathrm{I}_q^\zeta \mu(t) = \mu(t), \tag{3}$$

for each $t \in [0,\beta]$, where $\varrho = 1, \ldots, n-1$ and $[\zeta] = n-1$.

Lemma 2. *Let $\zeta \in (1,2]$, $0<\lambda, q<1$, ν_ϱ are real constants for $1 \le \varrho \le m$, $m \ge 2$ and $g \in C([0,\beta],\mathbb{R})$ be a given function. Then the unique solution of*

$$\begin{cases} {}^C\mathrm{D}_q^\zeta y(t) = g(t), \ 0<q<1, \ t \in [0,\beta], \\ y(0)=0, \quad \mathrm{D}_q y(\beta) = \sum_{\varrho=1}^{m} \nu_\varrho \mathrm{I}_q^{\zeta-1} y(\eta_\varrho), \ 0<\eta_\varrho<\beta, \end{cases} \tag{4}$$

is given by

$$\begin{aligned} y(t) = {} & \int_0^t \frac{(t-qs)^{(\zeta-1)}}{\Gamma_q(\zeta)} g(s)\mathrm{d}_q s + t\sum_{\varrho=1}^{m} \nu_\varrho \int_0^{\eta_\varrho} \frac{(\eta_\varrho - qs)^{(\zeta-1)}}{\Gamma_q(\zeta-1)} g(s)\mathrm{d}_q s \\ & - t\int_0^\beta \frac{(\beta-qs)^{(\zeta-2)}}{\Gamma_q(\zeta-1)} g(s)\mathrm{d}_q s. \end{aligned} \tag{5}$$

Proof. Applying I_q^ζ on (4), we get

$$y(t) = \int_0^t \frac{(t-qs)^{(\zeta-1)}}{\Gamma_q(\zeta)} g(s)\mathrm{d}_q s + c_0 + c_1 t, \tag{6}$$

for some constants $c_0, c_1 \in \mathbb{R}$. Since $y(0)=0$, we have $c_0 = 0$ and

$$\mathrm{D}_q y(t) = \mathrm{I}_q^{\zeta-1} g(t) + c_1.$$

From $\mathrm{D}_q y(\beta) = \sum_{\varrho=1}^{m} \nu_\varrho \mathrm{I}_q^{\zeta-1} y(\eta_\varrho)$, we have

$$c_1 = \sum_{\varrho=1}^{m} \frac{\nu_\varrho}{\Gamma_q(\zeta-1)} \int_0^{\eta_\varrho} (\eta_\varrho - qs)^{(\zeta-2)} g(s)\mathrm{d}_q s - \int_0^\beta \frac{(\beta-qs)^{(\zeta-2)}}{\Gamma_q(\zeta-1)} g(s)\mathrm{d}_q s.$$

So,

$$\begin{aligned} y(t) = {} & \int_0^t \frac{(t-qs)^{(\zeta-1)}}{\Gamma_q(\zeta)} g(s)\mathrm{d}_q s + t\sum_{\varrho=1}^{m} \nu_\varrho \int_0^{\eta_\varrho} \frac{(\eta_\varrho - qs)^{(\zeta-2)}}{\Gamma_q(\zeta-1)} g(s)\mathrm{d}_q s \\ & - t\int_0^\beta \frac{(\beta-qs)^{(\zeta-2)}}{\Gamma_q(\zeta-1)} g(s)\mathrm{d}_q s. \end{aligned}$$

Conversely, let us now demonstrate that if (5) satisfies (4). Applying operator ${}^C\mathrm{D}_q^\zeta$ on both sides of (5), then, from Lemma 1 we obtain

$${}^C\mathrm{D}_q^\zeta \mu(t) = g(t). \tag{7}$$

Taking the limit $t \to 0$ of (5) we obtain

$$y(0) = 0. \tag{8}$$

Now, Applying D_q and Definition 3 to both sides of (5) gives

$$\mathrm{D}_q y(t) = \mathrm{I}_q^{\zeta-1} g(t) + \sum_{\varrho=1}^{m} \nu_\varrho \mathrm{I}_q^{\zeta-1} y(\eta_\varrho) - \mathrm{I}_q^{\zeta-1} g(t). \tag{9}$$

Taking the limit $t \to \beta$ of (9) we have

$$\mathrm{D}_q y(\beta) = \sum_{\varrho=1}^{m} \nu_\varrho \mathrm{I}_q^{\zeta-1} y\left(\eta_\varrho\right).$$

Lastly, it is clear that the function in (5) meets the associated boundary conditions. □

3. Main Results

We denote by $\mathbb{F} = C\left([0,\beta], \mathbb{R}\right)$ the Banach space of all continuous functions from $[0,\beta]$ to $\mathbb{R}$ with

$$\|y\| = \sup_{t\in[0,\beta]} |y(t)|.$$

By Lemma 2, we define $N : \mathbb{F} \to \mathbb{F}$ by:

$$\begin{aligned} Ny(t) := & \int_0^t \frac{(t-qs)^{(\zeta-1)}}{\Gamma_q(\zeta)} f(s, y(s), y(\lambda s))\mathrm{d}_q s \\ & + \frac{t\sum_{\varrho=1}^{m} \nu_\varrho}{\Gamma_q(\zeta-1)} \int_0^{\eta_\varrho} (\eta_\varrho - qs)^{(\zeta-2)} f(s, y(s), y(\lambda s))\mathrm{d}_q s \\ & - t\int_0^\beta \frac{(\beta-qs)^{(\zeta-2)}}{\Gamma_q(\zeta-1)} f(s, y(s), y(\lambda s))\mathrm{d}_q s. \end{aligned}$$

Let

$$\Lambda = \frac{\beta^\zeta}{\Gamma_q(\zeta+1)} + \beta\left(\frac{m \max\limits_{1\le\varrho\le m} |\nu_\varrho| \sum\limits_{\varrho=1}^{m} \eta_\varrho^{\zeta-1}}{\Gamma_q(\zeta)}\right) + \frac{\beta^{\zeta-1}}{\Gamma_q(\zeta)}. \tag{10}$$

Now, we present the existence and uniqueness of solutions of (1) by using Banach's fixed point theorem.

Theorem 1. *Let $f : [0,\beta] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying the hypothesis*

($\mathcal{H}1$) *There exists $\varpi_i > 0$, $i = 1, 2$, such that for all $t \in [0,\beta]$ and all $y, \bar{y}, z, \bar{z} \in \mathbb{R}$, we have*

$$|f(t, y, \bar{y}) - f(t, z, \bar{z})| \le \varpi_1 |y - z| + \varpi_2 |\bar{y} - \bar{z}|.$$

There exists $t_o \in [0,\beta]$ such that $f(t_0, 0, 0) \neq 0$. Then the multi-point boundary value problem (1) has a unique solution if

$$\varpi\Lambda < 1,$$

where $\varpi = \varpi_1 + \varpi_2$.

Proof. Let for y, $z \in \mathbb{F}$ and for any $t \in [0,\beta]$, we get

$$
\begin{aligned}
&\|Ny - Nz\| \\
&\le \sup_{t\in[0,\beta]} \Bigg\{ \int_0^t \frac{(t-qs)^{(\zeta-1)}}{\Gamma_q(\zeta)} |f(s,y(s),y(\lambda s)) - f(s,z(s),z(\lambda s))| \mathrm{d}_q s \\
&\quad + t \Bigg(\sum_{\varrho=1}^{m} \frac{|\nu_\varrho|}{\Gamma_q(\zeta-1)} \int_0^{\eta_\varrho} (\eta_\varrho - qs)^{(\zeta-2)} |f(s,y(s),y(\lambda s)) - f(s,z(s),z(\lambda s))| \mathrm{d}_q s \\
&\quad + \int_0^{\beta} \frac{(\beta-qs)^{(\zeta-2)}}{\Gamma_q(\zeta-1)} |f(s,y(s),y(\lambda s)) - f(s,z(s),z(\lambda s))| \mathrm{d}_q s \Bigg) \Bigg\} \\
&\le \sup_{t\in[0,\beta]} \Bigg\{ \int_0^t \frac{(t-qs)^{(\zeta-1)}}{\Gamma_q(\zeta)} \mathrm{d}_q s + t \Bigg(\sum_{\varrho=1}^{m} \frac{|\nu_\varrho|}{\Gamma_q(\zeta-1)} \int_0^{\eta_\varrho} (\eta_\varrho - qs)^{(\zeta-2)} \mathrm{d}_q s \\
&\quad - \int_0^{\beta} \frac{(\beta-qs)^{(\zeta-2)}}{\Gamma_q(\zeta-1)} \mathrm{d}_q s \Bigg) \Bigg\} \varpi \|y-z\| \\
&\le \left[\frac{\beta^\zeta}{\Gamma_q(\zeta+1)} + \beta \left(\sum_{\varrho=1}^{m} |\nu_\varrho| \frac{\eta_\varrho^{\zeta-1}}{\Gamma_q(\zeta)} \right) + \frac{\beta^{\zeta-1}}{\Gamma_q(\zeta)} \right] \varpi \|y-z\| \\
&= \varpi\Lambda \|y-z\|,
\end{aligned}
$$

which leads to $\|Ny - Nz\| \le \varpi\Lambda \|y-z\|$. Since $\varpi\Lambda < 1$, N is a contraction by utilizing Banach's fixed theorem, the BVP (1) has a unique solution. □

Also, we give another variant of existence and uniqueness result based on the Hölder inequality.

Theorem 2. *Let $f : [0,\beta] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. In addition we assume that:*

($\mathcal{H}2$) $|f(t,y,\bar{y}) - f(t,z,\bar{z})| \le u(t)|y-z| + v(t)|\bar{y}-\bar{z}|$, $t \in [0,\beta]$ *and* $y, \bar{y}, z, \bar{z} \in \mathbb{R}$,

where $u, v \in L^{\frac{1}{\delta}}([0,\beta], \mathbb{R}^+)$ and $\delta \in (0,1)$. There exists $t_o \in [0,\beta]$ such that $f(t_0,0,0) \neq 0$.

Denote $\|\theta\|_{L^{\frac{1}{\delta}}} = \left(\int_0^\beta |\theta(s)|^{\frac{1}{\delta}} \mathrm{d}_q s \right)^\delta$, $\theta = u, v$.

If

$$(\|u\|_{L^{\frac{1}{\delta}}} + \|v\|_{L^{\frac{1}{\delta}}})\Delta < 1, \tag{11}$$

where

$$\Delta = \frac{1}{\Gamma_q(\zeta)} \left(\int_0^\beta (\beta - qs)^{\frac{(\zeta-1)}{1-\delta}} \mathrm{d}_q s \right)^{1-\delta}$$

$$+\beta\left[\frac{\sum_{\varrho=1}^{m}|\nu_\varrho|}{\Gamma_q(\zeta-1)}\left(\int_0^{\eta_\varrho}(\eta_\varrho-qs)^{\frac{(\zeta-2)}{1-\delta}}\mathrm{d}_qs\right)^{1-\delta}\right.$$

$$\left.+\frac{1}{\Gamma_q(\zeta-1)}\left(\int_0^{\beta}(\beta-qs)^{\frac{(\zeta-2)}{1-\delta}}\mathrm{d}_qs\right)^{1-\delta}\right]. \tag{12}$$

Then the multi-point boundary value problem (1) *has a unique solution.*

Proof. For $y,z\in\mathbb{F}$ and $t\in[0,\beta]$, by Hölder inequality and using $(\mathcal{H}2)$, we have:

$$\begin{aligned}
&\|Ny-Nz\| \\
&\leq \sup_{t\in[0,\beta]}\left\{\int_0^t\frac{(t-qs)^{(\zeta-1)}}{\Gamma_q(\zeta)}u(s)|y(s)-z(s)|\mathrm{d}_qs+\int_0^t\frac{(t-qs)^{(\zeta-1)}}{\Gamma_q(\zeta)}v(s)|y(s)-z(s)|\mathrm{d}_qs\right.\\
&\quad+t\left(\sum_{\varrho=1}^{m}\frac{|\nu_\varrho|}{\Gamma_q(\zeta-1)}\int_0^{\eta_\varrho}(\eta_\varrho-qs)^{(\zeta-2)}u(s)|y(s)-z(s)|\mathrm{d}_qs\right.\\
&\quad+\sum_{\varrho=1}^{m}\frac{|\nu_\varrho|}{\Gamma_q(\zeta-1)}\int_0^{\eta_\varrho}(\eta_\varrho-qs)^{(\zeta-2)}v(s)|y(s)-z(s)|\mathrm{d}_qs\\
&\quad\left.\left.+\int_0^{\beta}\frac{(\beta-qs)^{(\zeta-2)}}{\Gamma_q(\zeta-1)}v(s)|y(s)-z(s)|\mathrm{d}_qs+\int_0^{\beta}\frac{(\beta-qs)^{(\zeta-2)}}{\Gamma_q(\zeta-1)}u(s)|y(s)-z(s)|\mathrm{d}_qs\right)\right\}\\
&\leq \sup_{t\in[0,\beta]}\left\{\frac{1}{\Gamma_q(\zeta)}\left(\int_0^t(t-qs)^{\frac{(\zeta-1)}{1-\delta}}\mathrm{d}_qs\right)^{1-\delta}\left[\left(\int_0^t u(s)^{\frac{1}{\delta}}\mathrm{d}_qs\right)^{\delta}+\left(\int_0^t v(s)^{\frac{1}{\delta}}\mathrm{d}_qs\right)^{\delta}\right]\right.\\
&\quad+t\left(\sum_{\varrho=1}^{m}\frac{|\nu_\varrho|}{\Gamma_q(\zeta-1)}\left(\int_0^{\eta_\varrho}(\eta_\varrho-qs)^{\frac{(\zeta-2)}{1-\delta}}\mathrm{d}_qs\right)^{1-\delta}\right.\\
&\quad\times\left[\left(\int_0^{\eta_\varrho}u(s)^{\frac{1}{\delta}}\mathrm{d}_qs\right)^{\delta}+\left(\int_0^{\eta_\varrho}v(s)^{\frac{1}{\delta}}\mathrm{d}_qs\right)^{\delta}\right]\\
&\quad+\frac{1}{\Gamma_q(\zeta-1)}\left(\int_0^{\beta}(\beta-qs)^{\frac{(\zeta-2)}{1-\delta}}\mathrm{d}_qs\right)^{1-\delta}\\
&\quad\left.\left.\times\left[\left(\int_0^{\beta}u(s)^{\frac{1}{\delta}}\mathrm{d}_qs\right)^{\delta}+\left(\int_0^{\beta}v(s)^{\frac{1}{\delta}}\mathrm{d}_qs\right)^{\delta}\right]\right)\right\}\times\|y-z\|\\
&=\Delta(\|u\|_{L^{\frac{1}{\delta}}}+\|v\|_{L^{\frac{1}{\delta}}})\|y-z\|.
\end{aligned}$$

Therefore,

$$\|Ny-Nz\|\leq\Delta(\|u\|_{L^{\frac{1}{\delta}}}+\|v\|_{L^{\frac{1}{\delta}}})\|y-z\|.$$

By (11), N is a contraction mapping. Hence, (1) has a unique solution. □

Now, we prove the existence of solutions of multi-point boundary value problem (1) by applying Leray-Schauder nonlinear alternative [35].

Theorem 3. *Assume that $f : [0, \beta] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. There exists $t_o \in [0, \beta]$ such that $f(t_0, 0, 0) \neq 0$. Suppose that:*

($\mathcal{H}3$) *There exists a nondecreasing function $\psi : [0, \infty) \to (0, \infty)$, and a function $b \in C([0, \beta], \mathbb{R}^+)$, such that*

$$|f(t, y, \bar{y})| \leq b(t)\, \psi(|y|), \text{ for all } (t, y, \bar{y}) \in [0, \beta] \times \mathbb{R}^2.$$

($\mathcal{H}4$) *There exists $\mathfrak{Z} > 0$ such that*

$$\frac{\mathfrak{Z}}{\Lambda \|b\|\, \psi(\mathfrak{Z})} > 1,$$

where Λ given by (10). Then (1) has at least one solution on $[0, \beta]$.

Proof. The proof will be given in several steps.
Step 1: We demonstrate that the operator $N : \mathbb{F} \to \mathbb{F}$ be defined by (10) is completely continuous on $\mathbb{F}$. To achieve this, we first establish that the operator N is continuous on $\mathbb{F}$. Let $\{y_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathbb{F}$ that converges to a point $y \in \mathbb{F}$. Applying the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{n \to +\infty} Ny_n(t) := \lim_{n \to +\infty} \Bigg\{ & \int_0^t \frac{(t - qs)^{(\zeta - 1)}}{\Gamma_q(\zeta)} f(s, y_n(s), y_n(\lambda s)) \mathrm{d}_q s \\ & + \frac{t \sum_{\varrho=1}^{m} \nu_\varrho}{\Gamma_q(\zeta - 1)} \int_0^{\eta_\varrho} (\eta_\varrho - qs)^{(\zeta - 2)} f(s, y_n(s), y_n(\lambda s)) \mathrm{d}_q s \\ & - t \int_0^\beta \frac{(\beta - qs)^{(\zeta - 2)}}{\Gamma_q(\zeta - 1)} f(s, y_n(s), y_n(\lambda s)) \mathrm{d}_q s \Bigg\} \\ & = Ny(t). \end{aligned} \tag{13}$$

For all $t \in [0, \beta]$, we obtain

$$|Ny_n(t) - Ny(t)| \longrightarrow 0, \text{ as } n \longrightarrow 0.$$

Hence,

$$\|Ny_n - Ny\|_{\mathbb{F}} \longrightarrow 0, \text{ as } n \longrightarrow 0.$$

This shows that N is a continuous operator on $\mathbb{F}$.
Step 2: We will show that N maps bounded sets into bounded sets in $\mathbb{F}$. Let us define $L = \sup_{t \in [0, \beta]} |f(t, 0, 0)|$. Setting $\gamma \geq \frac{\Lambda L}{1 - \varpi \Lambda}$ with $0 \leq \varpi \Lambda < 1$, we show that $N\mathcal{B}_\gamma \subset \mathcal{B}_\gamma$, where $\mathcal{B}_\gamma = \{y \in \mathbb{F} : \|y\| \leq \gamma\}$. For $y \in \mathcal{B}_\gamma$ and for each $t \in [0, \beta]$, from the definition of N and hypothesis ($\mathcal{H}1$), we obtain

$$\|Ny\| \leq \sup_{t \in [0, \beta]} \Bigg\{ \int_0^t \frac{(t - qs)^{(\zeta - 1)}}{\Gamma_q(\zeta)} |f(s, y(s), y(\lambda s))| \mathrm{d}_q s$$

$$+ t \left(\sum_{\varrho=1}^{m} \frac{|\nu_\varrho|}{\Gamma_q(\zeta-1)} \int_0^{\eta_\varrho} (\eta_\varrho - qs)^{(\zeta-2)} |f(s,y(s),y(\lambda s))| \mathrm{d}_q s \right.$$
$$\left. + \int_0^{\beta} \frac{(\beta - qs)^{(\zeta-2)}}{\Gamma_q(\zeta-1)} |f(s,y(s),y(\lambda s))| \mathrm{d}_q s \right) \Bigg\}.$$
$$\leq \sup_{t\in[0,\beta]} \left\{ \int_0^t \frac{(t-qs)^{(\zeta-1)}}{\Gamma_q(\zeta)} (|f(s,y(s),y(\lambda s)) - f(s,0,0)| + |f(s,0,0)|) \mathrm{d}_q s \right.$$
$$+ t \left(\sum_{\varrho=1}^{m} \frac{|\nu_\varrho|}{\Gamma_q(\zeta-1)} \int_0^{\eta_\varrho} (\eta_\varrho - qs)^{(\zeta-2)} (|f(s,y(s),y(\lambda s)) - f(s,0,0)| \right.$$
$$+ |f(s,0,0)|) \mathrm{d}_q s$$
$$\left.\left. + \int_0^{\beta} \frac{(\beta - qs)^{(\zeta-2)}}{\Gamma_q(\zeta-1)} (|f(s,y(s),y(\lambda s)) - f(s,0,0)| + |f(s,0,0)|) \mathrm{d}_q s \right) \right\}$$
$$\leq (\varpi\wp + L) \sup_{t\in[0,\beta]} \left\{ \int_0^t \frac{(t-qs)^{(\zeta-1)}}{\Gamma_q(\zeta)} \mathrm{d}_q s \right.$$
$$\left. + t \left(\sum_{\varrho=1}^{m} \frac{|\nu_\varrho|}{\Gamma_q(\zeta-1)} \int_0^{\eta_\varrho} (\eta_\varrho - qs)^{(\zeta-2)} \mathrm{d}_q s - \int_0^{\beta} \frac{(\beta - qs)^{(\zeta-2)}}{\Gamma_q(\zeta-1)} \mathrm{d}_q s \right) \right\}$$
$$\leq (\varpi\wp + L) \left[\frac{\beta^\zeta}{\Gamma_q(\zeta+1)} + \beta \left(\sum_{\varrho=1}^{m} |\nu_\varrho| \frac{\eta_\varrho^{\zeta-1}}{\Gamma_q(\zeta)} \right) + \frac{\beta^{\zeta-1}}{\Gamma_q(\zeta)} \right]$$
$$= (\varpi\gamma + L)\Lambda \leq \gamma,$$

which implies that $N\mathcal{B}_\gamma \subset \mathcal{B}_\gamma$. Thus, N is uniformly bounded on $\mathcal{B}_\gamma$.

Step 3: We show that N maps bounded sets into equicontinuous sets of $\mathbb{F}$. Let $t_1, t_2 \in [0,\beta]$ with $t_1 < t_2$, $y \in \mathcal{B}_\gamma$ and by using hypothesis $(\mathcal{H}3)$, we have

$$|Ny(t_2) - Ny(t_1)|$$
$$\leq \frac{1}{\Gamma_q(\zeta)} \int_0^{t_1} \left((t_2 - qs)^{\zeta-1} - (t_1 - qs)^{\zeta-1} \right) |f(s,y(t)(s),y(t)(\lambda s))| \mathrm{d}_q s$$
$$+ \frac{1}{\Gamma_q(\zeta)} \int_{t_1}^{t_2} (t_2 - qs)^{\zeta-1} |f(s,y(t)(s),y(t)(\lambda s))| \mathrm{d}_q s$$
$$+ (t_2 - t_1) \left(\sum_{\varrho=1}^{m} \frac{|\nu_\varrho|}{\Gamma_q(2\zeta-1)} \int_0^{\eta_\varrho} (\eta_\varrho - qs)^{(2\zeta-2)} |f(s,y(s),y(\lambda s))| \mathrm{d}_q s \right.$$
$$\left. + \int_0^{\beta} \frac{(\beta - qs)^{(2\zeta-3)}}{\Gamma_q(2\zeta-2)} |f(s,y(s),y(\lambda s))| \mathrm{d}_q s \right)$$
$$\leq \frac{1}{\Gamma_q(\zeta)} \int_0^{t_1} [(t_2 - qs)^{\zeta-1} - (t_1 - qs)^{\zeta-1}] b(s)\, \psi(\wp)\, \mathrm{d}_q s$$

$$
\begin{aligned}
&+\frac{1}{\Gamma_q(\zeta)}\int_{t_1}^{t_2}(t_2-qs)^{\zeta-1}b\,(s)\,\psi\,(\wp)\,\mathrm{d}_qs\\
&+(t_2-t_1)\left(\sum_{\varrho=1}^{m}\frac{|\nu_\varrho|}{\Gamma_q(\zeta-1)}\int_0^{\eta_\varrho}(\eta_\varrho-qs)^{(\zeta-2)}b(s)\psi(\wp)\mathrm{d}_qs\right.\\
&\left.+\int_0^{\beta}\frac{(\beta-qs)^{(\zeta-2)}}{\Gamma_q(\zeta-1)}b\,(s)\,\psi\,(\wp)\,\mathrm{d}_qs\right)\\
\leq{}&\|b\|\,\psi(\wp)\frac{2(t_2^{\zeta}-t_1^{\zeta})}{\Gamma_q(\zeta+1)}+(t_2-t_1)\left(\sum_{\varrho=1}^{m}\|b\|\,\psi(\wp)|\nu_\varrho|\frac{\eta_\varrho^{\zeta-1}}{\Gamma_q(\zeta)}\right.\\
&\left.+\frac{\beta^{\zeta-1}}{\Gamma_q(\zeta)}\|b\|\psi(\wp)\right).
\end{aligned}
$$

The right-hand side of the above inequality tends to zero independently of $y\in\mathcal{B}_\gamma$ as $t_2-t_1\to 0$. Therefore, $N:\mathbb{F}\to\mathbb{F}$ is completely continuous by application of the Arzelá-Ascoli theorem.

Step 4: Consider the equation $y=\sigma Ny$, for $0<\sigma<1$ and assume that y be a solution. Then,

$$\|y\|=\|\sigma Ny\|\leq\Lambda\,\|b\|\,\psi\,(\wp)\,.$$

Therefore,

$$\frac{\|y\|}{\Lambda\,\|b\|\,\psi\,(\wp)}\leq 1.$$

By $(\mathcal{H}4)$, there exists $\mathfrak{Z}$ such that $\mathfrak{Z}\neq\|y(t)\|$. Let us set

$$\Theta=\{y\in\mathbb{F}:\|y\|<\mathfrak{Z}\}\,.$$

We see that the operator $N:\overline{\Theta}\to\mathbb{F}$ is continuous and completely continuous. From the choice of Θ, there is no $y\in\partial\Theta$ such that $y=\sigma Ny$ for some $0<\sigma<1$. Consequently, we deduce that N has a fixed point $y\in\overline{\Theta}$ which is a solution of (1). □

We also prove the existence of solutions of multi-point boundary value problem (1) by using Leray-Schauder degree.

Theorem 4. *Let $f:[0,\beta]\times\mathbb{R}^2\to\mathbb{R}$ be a continuous function. Suppose that*

$(\mathcal{H}5)$ *There exist constants $0\leq a<\frac{1}{\Lambda}$ and $r>0$ such that*

$$|f(t,y,z)|\leq r+a_1|y|+a_2|z|,\ \ (t,y,z)\in[0,\beta]\times\mathbb{R}^2,$$

where $a=a_1+a_2$.

Then, the multi-point boundary value problem (1) has at least one solution on $[0,\beta]$.

Proof. We define an operator $N : \mathbb{F} \to \mathbb{F}$ as in (10) and consider the fixed point equation

$$y = Ny.$$

We shall prove that there exists a fixed point $y \in \mathbb{F}$ satisfying (1). It is sufficient to show that $\mathbb{F} : \overline{\Omega}_{\wp} \to \mathbb{F}$ satisfies

$$y \neq \delta N y, \ \forall (y, \delta) \in \partial\Omega_{\wp} \times [0,1], \tag{14}$$

where

$$\Omega_{\wp} := \left\{ y \in \mathbb{F} : \sup_{t\in[0,\beta]} |y(t)| < \wp, \ \wp > 0 \right\}.$$

We define

$$S(\mu, y(t)) = \delta N y(t), \ \ (y, \delta) \in \mathbb{F} \times [0,1].$$

As in Theorem 3, N is continuous, uniformly bounded, and equicontinuous. Then, by the Arzela-Ascoli theorem, a continuous map s_δ defined by $s_\delta = y(t) - S(\mu, y(t)) = y(t) - \delta N y(t)$ is completely continuous. If (14) holds, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned} \deg(s_\delta, \Omega_\wp, 0) &= \deg(I - \delta N, \Omega_\wp, 0) = \deg(s_1, \Omega_\wp, 0) = \deg(s_0, \Omega_\wp, 0) \\ &= \deg(I, \Omega_\wp, 0) = 1 \neq 0, \qquad 0 \in \Omega_\wp, \end{aligned}$$

where I denotes the identity operator. By the nonzero property of Leray-Schauder's degree, $s_1(y(t)) = y(t) - Ny(t) = 0$ for at least one $y \in \Omega_\wp$. In order to prove (14), we assume that $y(t) = \delta N y(t)$ for some $\delta \in [0,1]$ and for all $t \in [0, \beta]$. Then

$$\begin{aligned} |y(t)| &= |\delta N y(t)| \\ &\leq a\Lambda |y(t)| + r\Lambda. \end{aligned}$$

Taking norm $\|y\| = \sup_{t\in[0,\beta]} |y(t)|$, we get

$$\|y\| \leq a\Lambda \|y\| + r\Lambda,$$

which implies that

$$\|y\| \leq \frac{r\Lambda}{1 - a\Lambda}.$$

If $\wp = \frac{r\Lambda}{1-a\Lambda} + 1$, then inequality (14) holds. □

4. Applications

Example 1. Let us consider the following multi-point boundary value problem

$$\begin{cases} {}^{C}\mathrm{D}_{0.5}^{\frac{3}{2}} y(t) = f(t, y(t), y(\lambda t)), \ t \in [0,1], \\ y(0) = 0, \ \mathrm{D}_q y(1) = \sum_{\varrho=1}^{m} \nu_\varrho \mathrm{I}_{0.5}^{\frac{1}{2}} y(\eta_\varrho), \end{cases} \tag{15}$$

For this example, we have $\zeta = \frac{3}{2}, \beta = 1, q = \frac{1}{2}, \lambda = \frac{1}{2}, \nu_\varrho = 2, (\varrho = 1, 2), \eta_1 = \frac{1}{5}, \eta_2 = \frac{1}{3}$ and $f(t, y(t), y(\frac{t}{2})) = \frac{\cos(y(t)) + \sin(\frac{y(t)}{2})}{32\pi(te^{t^2}+1)}$. Also for $y, \bar{y}, z, \bar{z} \in \mathbb{R}$ and $t \in [0, 1]$, we have

$$|f(t, y, \bar{y}) - f(t, z, \bar{z})| \leq \frac{1}{32\pi} |y - z| + \frac{1}{32\pi} |\bar{y} - \bar{z}| .$$

Hence,

$$\varpi_1 = \frac{1}{32\pi}, \qquad \varpi_2 = \frac{1}{32\pi},$$

$$\Lambda = \frac{\beta^\zeta}{\Gamma_q(\zeta+1)} + \beta \left(\frac{m \max\limits_{1 \leq \varrho \leq m} |\nu_\varrho| \sum\limits_{\varrho=1}^{m} \eta_\varrho^{\zeta-1}}{\Gamma_q(\zeta)} \right) + \frac{\beta^{\zeta-1}}{\Gamma_q(\zeta)} \approx 4.9025,$$

and

$$\varpi = \varpi_1 + \varpi_2 \approx \frac{1}{16\pi} .$$

Therefore, we have

$$\varpi\Lambda \simeq 9.7532 \times 10^{-2} < 1$$

Hence, all the hypotheses of Theorem 1 are satisfied. Thus, by the conclusion of Theorem 1, the multi-point boundary value problem (15) has a unique solution.

Example 2. As a second illustrative example, let us take

$$\begin{cases} {}^{C}\mathrm{D}_{0.5}^{\frac{4}{3}} y(t) = f(t, y(t), \lambda y(t)), \ t \in [0, 1] , \\ y(0) = 0, \ \mathrm{D}_q y(1) = \sum\limits_{\varrho=1}^{m} \nu_\varrho \mathrm{I}_{0.5}^{\frac{1}{3}} y(t) (\eta_\varrho) . \end{cases} \tag{16}$$

Here, $\zeta = \frac{4}{3}$, $q = \frac{1}{2}$, $\beta = 1$, $\lambda = \frac{1}{5}$, $\Omega_1 = \sqrt{3}$, $\Omega_2 = \frac{2}{5}$, $\eta_1 = \frac{2}{11}$, $\eta_2 = \frac{1}{7}$ and

$$f(t, y(t), y(\tfrac{t}{5})) = e^t \frac{y^2(t) + \|y\| + 10^{-3}}{(1 + e^t)(15\|y\| + t + 7) + \tan(y(t)(\frac{t}{5}))}.$$

Then we can find that

$$\Lambda = \frac{\beta^\zeta}{\Gamma_q(\zeta+1)} + \beta \left(\frac{m \max\limits_{1 \leq \varrho \leq m} |\nu_\varrho| \sum\limits_{\varrho=1}^{m} \eta_\varrho^{\zeta-1}}{\Gamma_q(\zeta)} \right) + \frac{\beta^{\zeta-1}}{\Gamma_q(\zeta)} \approx 2.8276.$$

Clearly,

$$\left|f(t,y(t),y(\tfrac{t}{5}))\right| = \left|e^t \frac{\frac{y(t)^2}{3} + \|y\| + 10^{-3}}{(1+e^t)\,(15\,\|y\| + t + 7) + \tan(y(t)(\frac{t}{5}))}\right|$$
$$\leq \frac{e^t}{5\,(1+e^t)} \left(\frac{\|y\|}{3} + 1\right).$$

Choosing $b\,(t) = \frac{e^t}{5(1+e^t)}$ with $\|b\| = \frac{e}{5}$ and $\psi\,(\|y\|) = \frac{\|y\|}{3} + 1$, we can show that

$$\frac{\mathfrak{Z}}{\|b\|\left(\frac{\mathfrak{Z}}{3}+1\right)\Lambda} > 1,$$

which implies $\mathfrak{Z} > 3.1529$. Hence, by Theorem 3, the multi-point boundary value problem (16) has at least one solution on $[0,1]$.

5. Conclusion

In this paper, we address the existence and uniqueness of solutions of the multi-point boundary value problem of nonlinear fractional differential equations with two fractional derivatives. The existence of solutions is demonstrated by using a variety of fixed point theorems, including Banach's fixed point theorem, Leray Schauder's nonlinear alternative, and Leray-degree Schauder's theory. We provide several examples to demonstrate the validity of our results. In future studies, we can expand our work by incorporating several recently introduced fractional operators, including the ψ-Caputo or ψ-Hilfer with q-fractional derivatives, hybrid equations, impulsive arguments, problems with multipoint conditions and so much more.

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Houari Bouzid
Laboratory of Mathematics and Applications (LMA)
Department of Mathematics
Faculty of Exact Science and Informatics
Hassiba Benbouali University of Chlef
Ouled Fares, Chlef 02000
Algeria
E-mail address: hb.bouzid@univ-chlef.dz

Abdelkader Benali
Laboratory of Mathematics and Applications (LMA)
Department of Mathematics
Faculty of Exact Science and Informatics
Hassiba Benbouali University of Chlef
Ouled Fares, Chlef 02000
Algeria
E-mail address: benali4848@gmail.com

Abdelkrim Salim
Faculty of Technology
Hassiba Benbouali University of Chlef
P.O. Box 151 Chlef 02000
Algeria
E-mail address: salim.abdelkrim@yahoo.com